MATH 2020 Tutorial

$(r\cos\theta, r\sin\theta, z)$, $\qquad (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$\iiint dz\, r\,dr\,d\theta \qquad \iiint \rho^2\sin\phi\, d\rho\, d\phi\, d\theta.$

① $$\int_0^{\pi}\int_0^{\frac{\theta}{\pi}}\boxed{\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz}\, r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} \frac{1}{2}z^2\Big|_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} \frac{1}{2}\left(9(4-r^2)-(4-r^2)\right) r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 4(4-r^2)\, r\,dr\,d\theta.$$

$$= \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 16r - 4r^3\, dr\,d\theta.$$

$$= \int_0^{\pi} 8r^2 - r^4 \Big|_0^{\frac{\theta}{\pi}} d\theta$$

$$= \int_0^{\pi} 8\cdot\frac{\theta^2}{\pi^2} - \frac{\theta^4}{\pi^4}\, d\theta$$

$$= \frac{8}{\pi^2}\cdot\frac{\theta^3}{3} - \frac{\theta^5}{5\pi^4}\Big|_0^{\pi}$$

$$= \frac{8}{\pi^2}\cdot\frac{\pi^3}{3} - \frac{\pi^5}{5\pi^4} = \frac{8}{3}\pi - \frac{\pi}{5} = \frac{37}{15}\pi.$$

② Convert the integral
$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{x} (x^2+y^2)\,dz\,dx\,dy$$
to an equivalent integral in cylindrical coordinates and evaluate the result.

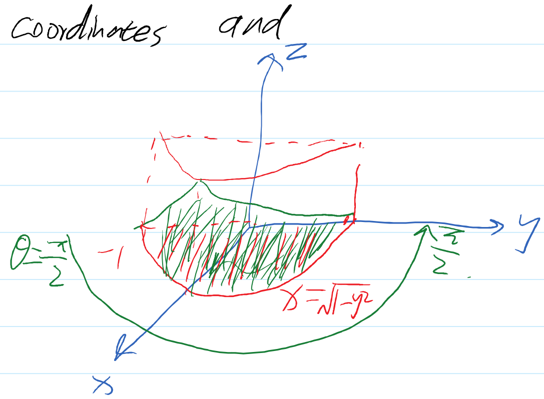
$(x,y)=(r\cos\theta, r\sin\theta)$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 \boxed{\int_0^{r\cos\theta} r^2\, dz}\, r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^3\cdot r\cos\theta\, dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cos\theta\cdot r^5\Big|_0^1 d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cos\theta\, d\theta = \frac{\sin\theta}{5}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{5}.$$



$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cos\theta \, d\theta = \frac{\sin\theta}{5}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2}{5}.$$

③ $$\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\boxed{\int_0^1 5\rho^3\sin^3\phi \, d\rho}\, d\phi \, d\theta.$$

$$= \int_0^{\frac{3}{2}\pi}\int_0^{\pi} \frac{5\sin^3\phi}{4}\rho^4\Big|_0^1 d\phi\, d\theta.$$

$$= \int_0^{\frac{3}{2}\pi}\int_0^{\pi} \frac{5\sin^3\phi}{4} d\phi\, d\theta.$$

$$= \int_0^{\frac{3}{2}\pi}\int_0^{\pi} \frac{5}{4}\sin\phi\,(1-\cos^2\phi)\, d\phi\, d\theta.$$

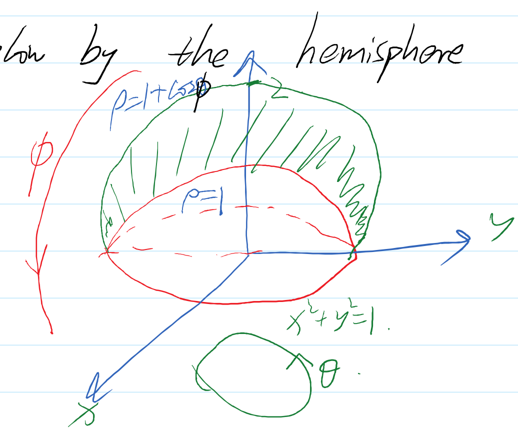
$$= \frac{5}{4}\int_0^{\frac{3}{2}\pi} -\cos\phi + \frac{1}{3}\cos^3\phi\Big|_0^{\pi} d\theta.$$

$$= \frac{5}{4}\int_0^{\frac{3}{2}\pi} \left(1-\frac{1}{3}\right) - \left(-1+\frac{1}{3}\right) d\theta.$$

$$= \frac{5}{4}\int_0^{\frac{3}{2}\pi} \frac{4}{3} d\theta = \frac{5}{4}\cdot\frac{4}{3}\cdot\left(\frac{3}{2}\pi - 0\right) = \frac{5}{2}\pi.$$

$(\cos^3\phi)'$
$= 3\cos^2\phi \cdot (-\sin\phi)$
$\int -\sin\phi\cos^2\phi\, d\phi$
$= \frac{1}{3}\cos^3\phi.$

④ Find the volumn of the solid bounded below by the hemisphere $\rho=1$, $z\geq 0$ and above by $\rho = 1+\cos\phi$.



$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\phi} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\cdot\frac{1}{3}\rho^3\Big|_1^{1+\cos\phi} d\phi\, d\theta.$$

$$= \frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\phi\,(1+\cos\phi)^3 - \sin\phi \, d\phi\, d\theta.$$

$$= \frac{1}{3}\int_0^{2\pi} -\frac{1}{4}(1+\cos\phi)^4 + \cos\phi\Big|_0^{\frac{\pi}{2}} d\theta.$$

$$= \frac{1}{3}\int_0^{2\pi} -\frac{1}{4} - \left(-\frac{1}{4}\cdot 2^4 + 1\right) d\theta$$

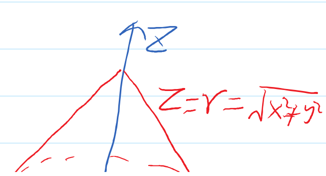
$$= \frac{1}{3}\int_0^{2\pi} -\frac{1}{4} - (-3)\, d\theta.$$

$$= \frac{1}{3}\times(2\pi - 0)\times\frac{11}{4} = \frac{11}{6}\pi.$$

$$\int \sin\phi\,(1+\cos\phi)^3 d\theta \qquad t = \cos\phi.$$

$$= -\frac{1}{4}(1+\cos\phi)^4$$

⑤ Find the volumn



$$\int^1 \int^{1-r} dz\, r\,dr\, d\theta$$

(5) Find the volume

$$\int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\, r\,dr\,d\theta$$

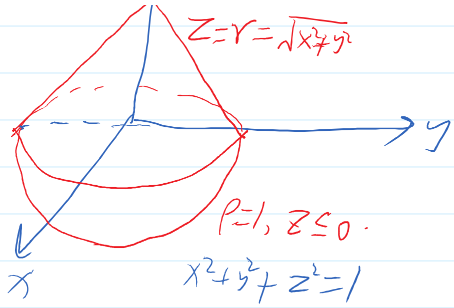
$$=\int_0^{2\pi}\int_0^1 (1-r+\sqrt{1-r^2})\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r-r^2+r(1-r^2)^{\frac12}\,dr\,d\theta.$$

$$=\int_0^{2\pi} \frac12 r^2-\frac13 r^3-\frac13(1-r^2)^{\frac32}\Big|_0^1 d\theta.$$

$$=\int_0^{2\pi}\left(\frac12-\frac13-0\right)-\left(-\frac13\right)d\theta.$$

$$=\int_0^{2\pi}\frac12\,d\theta=\frac12\times 2\pi=\pi.$$



$x^2+y^2+z^2=1$

$r^2+z^2=1$

$z=-\sqrt{1-r^2}.$

$\int r(1-r^2)^{\frac12}dr \quad t=1-r^2$

$=\int\sqrt{t}\cdot\left(-\frac12 dt\right). \quad dt=-2r\,dr$

⑥ A solid ball is bounded by the sphere $\rho=1$. Find the moment of inertia about the z-axis if the density is $\delta(\rho,\phi,\theta)=\rho^2$.

Ans.

$$I_z=\iiint \underbrace{(x^2+y^2)}\,\delta.$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\sin^2\phi\cdot\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^6\sin^3\phi\,d\rho\,d\phi\,d\theta.$$

$\sin^3\phi=\sin\phi\,(1-\cos^2\phi)$

$$=\int_0^{2\pi}\int_0^{\pi}\frac17\sin^3\phi\cdot\rho^7\Big|_0^1 d\phi\,d\theta.$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac17\sin^3\phi\,d\phi\,d\theta.$$

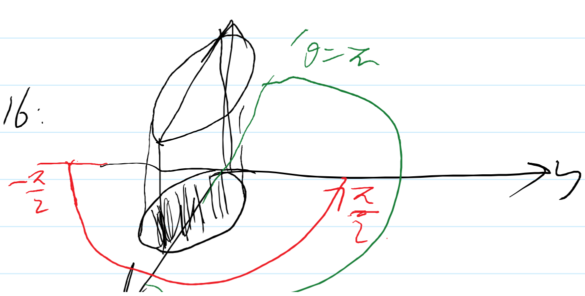
$$=\frac17\int_0^{2\pi}\int_0^{\pi}\sin\phi-\underbrace{\sin\phi\cos^2\phi}\,d\phi\,d\theta.$$

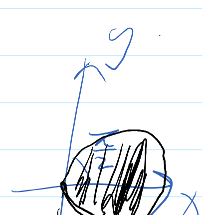
$$=\frac17\int_0^{2\pi}-\cos\phi+\frac13\cos^3\phi\Big|_0^{\pi}d\theta.$$

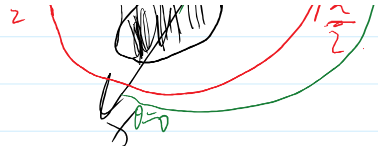
$$=\frac17\int_0^{2\pi}\left(1-\frac13\right)-\left(-1+\frac13\right)d\theta.$$

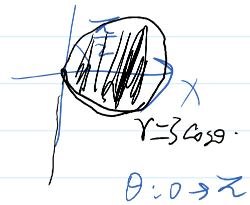
$$=\frac17\int_0^{2\pi}\frac43\,d\theta=\frac17\times\frac43\times(2\pi-0)=\frac{8}{21}\pi.\quad \#.$$

⑦ Q16:









$\theta: 0 \to \pi,\ \cos\pi = -1$

$r = -3$

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$(r\cos\theta,\ r\sin\theta,\ z)$ $\qquad$ $(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$.

$\iiint dz\, r\,dr\,d\theta$ $\qquad$ $\iiint \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

① $\displaystyle\int_0^{\pi}\int_0^{\frac{\theta}{\pi}}\boxed{\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz}\, r\,dr\,d\theta$

$\displaystyle = \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} \frac{1}{2}z^2\Big|_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} r\,dr\,d\theta.$

$\displaystyle = \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} \frac{1}{2}\left[9(4-r^2)-(4-r^2)\right] r\,dr\,d\theta$

$\displaystyle = \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 4(4-r^2)\,r\,dr\,d\theta$

$\displaystyle = \int_0^{\pi}\int_0^{\frac{\theta}{\pi}} 16r - 4r^3\,dr\,d\theta$

$\displaystyle = \int_0^{\pi} 8r^2 - r^4\Big|_0^{\frac{\theta}{\pi}} d\theta.$

$\displaystyle = \int_0^{\pi} 8\cdot\frac{\theta^2}{\pi^2} - \frac{\theta^4}{\pi^4}\,d\theta.$

$\displaystyle = \frac{8}{3\pi^2}\theta^3 - \frac{1}{5\pi^4}\theta^5\Big|_0^{\pi}$

$\displaystyle = \frac{8}{3\pi^2}\cdot\pi^3 - \frac{1}{5\pi^4}\cdot\pi^5 = \frac{8}{3}\pi - \frac{\pi}{5} = \frac{37}{15}\pi.$ #.

② Convert the integral

$$\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{x}(x^2+y^2)\,dz\,dx\,dy$$

to an equivalent integral in cylindrical coordinates and evalute the result.

$\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{1}\boxed{\int_0^{r\cos\theta}\boxed{r^2}\,dz}\,\boxed{r}\,dr\,d\theta$

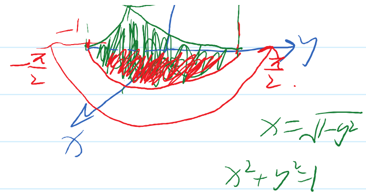
$\displaystyle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{1} r^3\cdot(r\cos\theta - 0)\,dr\,d\theta.$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^3\cdot(r\cos\theta-0)\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^4\cos\theta\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}r^5\cos\theta\Big|_0^1\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5}\cos\theta\,d\theta=\frac{1}{5}\sin\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{2}{5}\quad\#.$$



③ $$\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\int_0^1 5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\frac{5}{4}\rho^4\sin^3\phi\Big|_0^1\,d\phi\,d\theta$$

$$=\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\frac{5}{4}\sin^3\phi\,d\phi\,d\theta$$

$$=\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\frac{5}{4}\sin\phi\,\overbrace{(1-\cos^2\phi)}^{\sin^2\phi}\,d\phi\,d\theta$$

$\int\sin\phi\cos^2\phi\,d\phi$ $\quad t=\cos\phi$, $dt=-\sin\phi\,d\phi$

$$=\frac{5}{4}\int_0^{\frac{3}{2}\pi}\int_0^{\pi}\sin\phi-\sin\phi\cos^2\phi\,d\phi\,d\theta$$

$=-\int t^2\,dt$

$$=\frac{5}{4}\int_0^{\frac{3}{2}\pi}-\cos\phi+\frac{1}{3}\cos^3\phi\Big|_0^{\pi}\,d\theta$$

$$=\frac{5}{4}\int_0^{\frac{3}{2}\pi}\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)d\theta$$

$$=\frac{5}{4}\int_0^{\frac{3}{2}\pi}\frac{4}{3}\,d\theta=\frac{5}{4}\times\left(\frac{3}{2}\pi-0\right)\times\frac{4}{3}=\frac{5}{2}\pi\quad\#.$$

④ Find the volumn of the solid bounded below by the hemisphere $\rho=1,\ z\geq 0$ and above by $\boxed{\rho=1+\cos\phi}$.

Ans:

$$\underbrace{\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_1^{1+\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

$$=\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{1}{3}\rho^3\sin\phi\Big|_1^{1+\cos\phi}\,d\phi\,d\theta$$

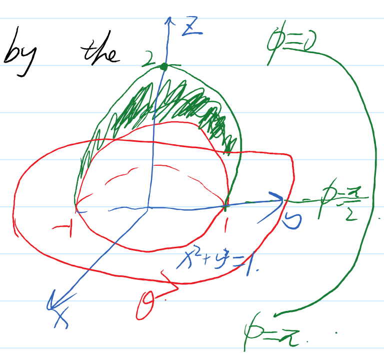
$$=\frac{1}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\sin\phi(1+\cos\phi)^3-\sin\phi\,d\phi\,d\theta$$

$\int\sin\phi(1+\cos\phi)^3\,d\phi$ $\quad t=\cos\phi$, $dt=-\sin\phi\,d\phi$

$=\int-(1+t)^3\,dt$

$$=\frac{1}{3}\int_0^{2\pi}-\frac{1}{4}(1+\cos\phi)^4+\cos\phi\Big|_0^{\frac{\pi}{2}}\,d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{4}+0\right)-\left(\underbrace{-\frac{1}{4}\cdot 2^4}+1\right)d\theta$$



$$=\frac{1}{3}\int_0^{2\pi}\left(-\frac{1}{4}+0\right)-\left(\underbrace{-\frac{1}{4}\cdot 2^4}_{-3}+1\right)d\theta$$

$$=\frac{1}{3}\int_0^{2\pi}\frac{11}{4}\,d\theta=\frac{1}{3}\times(2\pi-0)\times\frac{11}{4}=\frac{11}{6}\pi \quad \#.$$

⑤ Find the volumn

$$\int_0^{2\pi}\int_0^1\boxed{\int_{-\sqrt{1-r^2}}^{1-r} dz}\, r\,dr\,d\theta$$

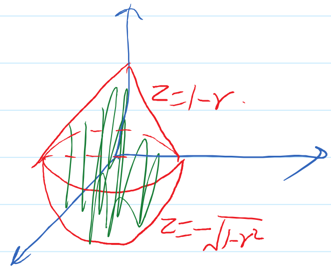
$$=\int_0^{2\pi}\int_0^1(1-r+\sqrt{1-r^2})r\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r-r^2+r(1-r^2)^{\frac{1}{2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\frac{1}{2}r^2-\frac{1}{3}r^3-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\Big|_0^1\,d\theta.$$

$$=\int_0^{2\pi}\left(\frac{1}{2}-\cancel{\frac{1}{3}}^{1}-0\right)-\left(-\cancel{\frac{1}{3}}^{0}\right)d\theta$$

$$=\int_0^{2\pi}\frac{1}{2}\,d\theta=\frac{1}{2}\times 2\pi=\pi.$$



$$\int r(1-r^2)^{\frac{1}{2}}dr \qquad t=1-r^2$$
$$=\int-\frac{1}{2}\sqrt{t}\,dt. \qquad dt=-2r\,dr$$

⑥ A solid ball is bounded by the sphere $\rho=1$
Find the moment of inertia about the z-axis if the density is $\delta(\rho,\phi,\theta)=\rho^2$

Ans: By definition,

$$(x,y,z)=(\rho\sin\phi\cos\theta,\rho\sin\phi\sin\theta,\rho\cos\phi)$$

$$I_z=\iiint(x^2+y^2)\,\delta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1\rho^2\sin^2\phi\cdot\rho^2\cdot\boxed{\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1\rho^6\sin^3\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac{1}{7}\rho^7\sin^3\phi\Big|_0^1\,d\phi\,d\theta.$$

$$=\int_0^{2\pi}\int_0^{\pi}\frac{1}{7}\sin^3\phi\,d\phi\,d\theta.$$

$$=\frac{1}{7}\int_0^{2\pi}\int_0^{\pi}\sin\phi(1-\cos^2\phi)\,d\phi\,d\theta.$$

$$=\frac{1}{7}\int_0^{2\pi}\int_0^{\pi}\sin\phi-\sin\phi\cos^2\phi\,d\phi\,d\theta.$$

$$=\frac{1}{7}\int_0^{2\pi}-\cos\phi+\frac{1}{3}\cos^3\phi\Big|_0^{\pi}\,d\theta.$$

$$= \frac{1}{7} \int_0^{2\pi} (1-\frac{1}{3}) - (-1+\frac{1}{3})\, d\theta.$$

$$= \frac{1}{7} \int_0^{2\pi} \frac{4}{3}\, d\theta = \frac{1}{7} \times (2\pi - 0) \times \frac{4}{3} = \frac{8}{21}\pi. \quad \#.$$